chinaXiv:202107.00011v3

Properties of several fuzzy set spaces ☆

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Abstract

This paper discusses the properties the spaces of fuzzy sets in a metric space equipped with the endograph metric and the sendograph metric, respectively. We fist discuss the level characterizations of the Γ-convergence and the endograph metric, and point out the elementary relationships among Γ-convergence, endograph metric and the sendograph metric. On the basis of these results, we present the characterizations of total boundedness, relative compactness and compactness in the space of compact positive α-cuts fuzzy sets equipped with the endograph metric, and in the space of compact support fuzzy sets equipped with the sendograph metric, respectively. Furthermore, we give completions of these two kinds of spaces, respectively.

Keywords: Endograph metric; Sendograph metric; Hausdorff metric; Total boundedness; Relative compactness; Compactness; Completion

0. Basic notions

In this section, we recall some basic notions related to fuzzy sets and convergence structures on them which will be discussed in this paper. Readers can refer to [2, 22] for more contents.

Let (X, d) be a metric space and let $K(X)$ and $C(X)$ denote the set of all non-empty compact subsets of X and the set of all non-empty closed subsets of X, respectively.

Let $F(X)$ denote the set of all fuzzy sets in X. A fuzzy set $u \in F(X)$ can be seen as a function $u : X \to [0, 1]$. In this sense, a subset S of X can

☆Project supported by Natural Science Foundation of Fujian Province of China(No. 2020J01706)

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be seen as a fuzzy set $S_{F(X)}$ in X

$$S_{F(X)}(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For $x \in X$, we use $\widehat{x}_X$ to denote the fuzzy set $\{x\}_{F(X)}$ in X. If there is no confusion, we will write $\widehat{x}_X$ as $\widehat{x}$ for simplicity.

For $u \in F(X)$, let $[u]_\alpha$ denote the α-cut of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0,1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the closure of S in (X, d).

For $u \in F(X)$, define

$$\operatorname{end} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\},$$
$$\operatorname{send} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\} \cap ([u]_0 \times [0,1]).$$

$\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph and the sendograph of u, respectively.

Let $F_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0,1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0,1]\}.$$

We introduce some subclasses of $F_{USC}(X)$, which will be discussed in this paper. Define

$$F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X)\},$$
$$F_{USCG}(X) := \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0,1]\}.$$

Clearly,

$$F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USC}(X).$$

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. The set of (compact) fuzzy numbers are denoted by E^m. It is defined as

$$E^m := \{u \in F_{USCB}(\mathbb{R}^m) : [u]_\alpha \text{ is a convex subset of } \mathbb{R}^m \text{ for } \alpha \in [0,1]\}.$$

Fuzzy numbers have attracted much attention from theoretical research and practical applications [1, 2, 5, 8, 20, 22].

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Let (X, d) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff metric*** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V),\ H^*(V,U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u\in U} d\,(u,V) = \sup_{u\in U}\inf_{v\in V} d\,(u,v).$$

The metric $\overline{d}$ on $X \times [0,1]$ is defined as

$$\overline{d}((x,\alpha),(y,\beta)) = d(x,y) + |\alpha - \beta|.$$

If there is no confusion, we also use H to denote the Hausdorff metric on $C(X \times [0,1])$ induced by $\overline{d}$.

We say that a sequence of sets $\{C_n\}$ ***Kuratowski converges*** to $C \subseteq X$, if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X :\ x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X :\ x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty}\overline{\bigcup_{m\geq n} C_m}.$$

In this case, we'll write $\boldsymbol{C = \lim_{n\to\infty} C_n(Kuratowski)}$ or $\boldsymbol{C = \lim_{n\to\infty} C_n(K)}$ for simplicity.

Rojas-Medar and Román-Flores [17] have introduced the Γ-convergence on $F_{USC}(X)$:

Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. Then u_n **Γ-converges** to u ($\boldsymbol{u_n \xrightarrow{\Gamma} u}$) if

$$\operatorname{end} u = \lim_{n\to\infty} \operatorname{end} u_n\,(K).$$

Let (X, d) be a metric space and let $u \in F_(X)$. Then

$$\begin{aligned} &u \text{ is upper semi-continuous} \\ &\quad\Leftrightarrow \operatorname{end} u \text{ is closed in } (X \times [0,1], \overline{d}) \\ &\quad\Leftrightarrow \operatorname{send} u \text{ is closed in } (X \times [0,1], \overline{d}). \end{aligned}$$

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The endograph metric H_{end} and the sendograph metric H_{send} can be defined on $F_{USC}(X)$ as usual. For $u, v \in F_{USC}(X)$,

$$\boldsymbol{H_{\text{end}}(u, v)} := H(\text{end}\, u, \text{end}\, v),$$
$$\boldsymbol{H_{\text{send}}(u, v)} := H(\text{send}\, u, \text{send}\, v).$$

The endograph metric H_{end} and the sendograph metric H_{send} are defined by using the Hausdorff metric on $C(X \times [0, 1])$ induced by $\overline{d}$ on $X \times [0, 1]$.

1. Introduction

A fuzzy set can be identified with its endograph. Also, a fuzzy set can be identified with its sendograph. So convergence structures on fuzzy sets can be defined on their endographs or sendographs. The Γ-convergence, the endograph metric H_{end} and the sendograh metric H_{send} are this kind of convergence structures. In this paper, we discuss the properties and the relations of these three convergence structures.

Compactness is one of the central concepts in topology and analysis and useful in applications (see [14, 21]). There is a lot of work devoted to characterizations of compactness in various fuzzy set spaces endowed with different topologies [3, 6, 7, 9, 10, 18, 19, 23].

The endograph metric is shown to has significant advantages [15, 16]. In [10], we presented the level characterizations of the Γ-convergence and the endograph metric H_{end}. Based on this, we have given the characterizations of total boundedness, relative compactness and compactness of fuzzy set spaces equipped with the endograph metric.

The results in [10] are obtained on the realm of fuzzy sets in $\mathbb{R}^m$. $\mathbb{R}^m$ is a special type of metric space. Of course, it is worth to study the fuzzy sets in metric space [6, 7, 13].

In this paper, we first discuss the level characterizations of the Γ-convergence and the endograph metric H_{end} on fuzzy sets in $F_{USC}(X)$. Based on this, we give the characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{end}})$, repectively. Here we mention that the characterization of relatively compact sets in $(F_{USCB}(X), H_{\text{send}})$ has already been given by Greco [6].

It is natural to consider what is a completion of a metric space. In this paper, we construct a completion of $(F_{USCB}(X), H_{\text{send}})$ and give a completion of $(F_{USCG}(X), H_{\text{end}})$, which is also a completion of $(F_{USCB}(X), H_{\text{end}})$.

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The remainder of this paper is organized as follows. In Section 2, we give elementary relationship among the Γ-convergence, the endograph metric and the sendoragraph metric. In Section 3, we investigate the level characterizations of the Γ-convergence. In Section 4, we consider the level characterizations of the endograph metric convergence. In Section 5, on the basis of the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{send}})$, respectively. In Section 6, we give completions of $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{send}})$, respectively. At last, we draw the conclusions in Section 7.

2. Elementary relationship among Γ-convergence, H_{end} and H_{send}

In this section, we give some conclusions which are useful in this paper. From some of these conclusions, we can obtain elementary relationship among Γ-convergence, H_{end} and H_{send}.

Theorem 2.1. *Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \ldots$. Then $H(C_n, C) \to 0$ implies that $\lim_{n\to\infty} C_n(K) = C$.*

Proof. This is an already known result. Its proof is similar to that of Theorem 4.1 in [10]. □

From Theorem 2.1, we know that if $H_{\text{end}}(u_n, u) \to 0$, then $u_n \xrightarrow{\Gamma} u$ for u, u_n, $n = 1, 2, \ldots$ in $F_{USC}(X)$.

Proposition 2.2. *Given u, u_n, $n = 1, 2, \ldots$ in $F_{USC}(X)$. Then*

(i) $H_{\text{send}}(u_n, u) \to 0$ *is equivalent to* $H_{\text{end}}(u_n, u) \to 0$ *and* $H([u_n]_0, [u]_0) \to 0$

(ii) $\lim_{n\to\infty} \operatorname{send} u_n(K) = \operatorname{send} u$ *is equivalent to* $u_n \xrightarrow{\Gamma} u$ *and* $\lim_{n\to\infty} [u_n]_0(K) = [u]_0$

The Hausdorff metric has the following important properties.

Theorem 2.3. *[18] Let (X, d) be a metric space and let H be the Hausdorff metric induced by d. Then the following statements are true.*
(i) (X, d) is complete $\Longleftrightarrow$ $(K(X), H)$ is complete.
(ii) (X, d) is separable $\Longleftrightarrow$ $(K(X), H)$ is separable.
(iii) (X, d) is compact $\Longleftrightarrow$ $(K(X), H)$ is compact.

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3. Level characterizations of Γ-convergence

In this section, we investigate the level characterizations of the Γ-convergence. It is found that the Γ-convergence has the level decomposition property on $F_{USCG}(X)$, fuzzy sets in which has compact positive α-cuts. It is pointed out that the Γ-convergence need not have the level decomposition property on $F_{USC}(X)$.

Rojas-Medar and Román-Flores [17] have introduced the following useful property of Γ-convergence.

Theorem 3.1. *[17] Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$. Then $u_n \xrightarrow{\Gamma} u$ iff for all $\alpha \in (0, 1]$,*

$$\{u > \alpha\} \subseteq \liminf_{n\to\infty}[u_n]_\alpha \subseteq \limsup_{n\to\infty}[u_n]_\alpha \subseteq [u]_\alpha. \tag{1}$$

Remark 3.2. Rojas-Medar and Román-Flores (Proposition 3.5 in [17]) presented the statement in Proposition 3.1 when u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in E^m. It can be checked that this conclusion also holds when u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$.

Theorem 3.3. *[10] Let (X, d) be a metric space and let $\{C_n\}$ be a sequence of sets in X. Then $\liminf_{n\to\infty} C_n$ and $\limsup_{n\to\infty} C_n$ are closed sets.*

Theorem 3.4. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$. Then $u_n \xrightarrow{\Gamma} u$ iff for all $\alpha \in (0, 1]$,*

$$\overline{\{u > \alpha\}} \subseteq \liminf_{n\to\infty}[u_n]_\alpha \subseteq \limsup_{n\to\infty}[u_n]_\alpha \subseteq [u]_\alpha.$$

Remark 3.5. Clearly, if $u_n \xrightarrow{\Gamma} u$, then $[u]_0 = \overline{\{u > 0\}} \subseteq \liminf_{n\to\infty}[u_n]_0$, and that $[u]_0 \subsetneqq \liminf_{n\to\infty}[u_n]_0$ could happen. By Proposition 2.2, $u_n \xrightarrow{\Gamma} u$ and $[u]_0 \supseteq \limsup_{n\to\infty}[u_n]_0$ if and only if $\lim_{n\to\infty} \operatorname{send} u_n(K) = \operatorname{send} u$.

Lemma 3.6. *[11] Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.*
(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $\bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
(ii) If $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots$ and $U = \overline{\bigcup_{n=1}^{+\infty} U_n} \in K(X)$, then $H(U_n, U) \to 0$ as $n \to +\infty$.

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Proof. This is Lemma 4.4 in [11]. Here we give a proof using Theorem 2.3. We only show (ii). (i) can be shown in a similar way.

Since $U_n \in K(U)$, then by Theorem 2.3, U_n has a subsequence which converges to $C \in K(U)$. Then clearly $H(U_n, C) \to 0$, and thus by Theorem 2.1, $C = U$.

□

Let u be a fuzzy set in $F_{USC}(X)$. Denote

- $P(u) := \{\alpha \in (0,1) : \overline{\{u > \alpha\}} \subsetneqq [u]_\alpha\}$.
- $P_0(u) := \{\alpha \in (0,1) : \lim_{\beta \to \alpha} H([u]_\beta, [u]_\alpha) \neq 0\}$.

A number α in $P(u)$ is called a platform point of u. Clearly, $P(u) \subseteq P_0(u)$. $P(u) \subsetneqq P_0(u)$ could happen. Consider $u \in F_{USC}(\mathbb{R}^2)$ given by

$$[u]_\alpha = \{z : \arg z \in [\alpha, 1]\} \cup \{0\} \text{ for each } \alpha \in [0,1]$$

where we write each $(x, y) \in \mathbb{R}^2$ as a complex number $z = x + iy$. Then $P(u) = \emptyset$ and $P_0(u) = (0,1)$.

From Lemma 3.6, we can obtain that $P(u) = P_0(u)$ for $u \in F_{USCG}(X)$. Combined with Lemma 6.12 in [11], we obtain the following conclusion

Lemma 3.7. *Given $u \in F_{USCG}(X)$. Then $P_0(u) = P(u)$ and $P(u)$ is at most countable.*

Theorem 3.8. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USC}(X)$. Then the following statements are true.*

(i) If $[u]_\alpha = \lim_{n\to\infty}[u_n]_\alpha\,(K)$ for $\alpha \in P$, where P is a dense set in $(0,1)$, then $u_n \xrightarrow{\Gamma} u$.

(ii) If $u_n \xrightarrow{\Gamma} u$, then $[u]_\alpha = \lim_{n\to\infty}[u_n]_\alpha\,(K)$ for all $\alpha \in (0,1) \setminus P(u)$.

Proof. The proof of (i) is similar to "(ii) ⇒ (i)" in the proof of Theorem 6.2 in [10]. (ii) follows immediately from Theorem 3.4. □

The following theorem states the level decomposition property of Γ-convergence on $F_{USCG}(X)$.

Theorem 3.9. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USCG}(X)$. Then the following statements are true.*

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(i) $u_n \xrightarrow{\Gamma} u$

(ii) $u_n \xrightarrow{a.e.} u\,(K)$.

(iii) $[u]_\alpha = \lim_{n\to\infty}[u_n]_\alpha\,(K)$ *for all* $\alpha \in (0,1) \setminus P(u)$

(iv) $\lim_{n\to\infty}[u_n]_\alpha\,(K) = [u]_\alpha$ *holds when* $\alpha \in P$, *where* P *is a dense subset of* $(0,1)\backslash P(u)$.

(v) $\lim_{n\to\infty}[u_n]_\alpha\,(K) = [u]_\alpha$ *holds when* $\alpha \in P$, *where* P *is a countable dense subset of* $(0,1)\backslash P(u)$

Proof. The desired results follow immediately from Lemma 3.7 and Theorem 3.8. □

Remark 3.10. It can be checked that the level decomposition property of Γ-convergence need not hold on $F_{USC}(X)$.

4. Level characterizations of endograph metric convergence

In this section, we discuss the level characterizations of endograph metric convergence.

Theorem 4.1. *Let* u, u_n, $n = 1, 2, \ldots$, *be fuzzy sets in* $F_{USC}(X)$ *and let* P *be a dense subset of* $[0,1]$. *Suppose that* $H([u_n]_\alpha, [u]_\alpha) \to 0$ *for each* $\alpha \in P$. *Then* $H_{\text{end}}(u_n, u) \to 0$.

Proof. We proceed by contradiction. If $H_{\text{end}}(u_n, u) \not\to 0$, then there is an $\varepsilon > 0$ such that $H_{\text{end}}(u_{n_k}, u) > \varepsilon$ for a subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Suppose $H^*(\text{end}\, u_{n_k}, \text{end}\, u) > \varepsilon$. Then there exists a sequence $(x_{n_k}, \alpha_{n_k}) \in \text{end}\, u_{n_k}$ such that

$$d((x_{n_k}, \alpha_{n_k}), \text{end}\, u) > \varepsilon. \tag{2}$$

With no loss of generality we can assume $\alpha_{n_k} \to \alpha \geq \varepsilon$. Pick $\beta \in P$ satisfying $\alpha \in (\beta, \beta + \varepsilon/2)$. Then there exists K such that $\alpha_{n_k} \in (\beta, \beta + \varepsilon/2)$ for all $k \geq K$. Thus for each $k \geq K$,

$$\begin{aligned} &d((x_{n_k}, \alpha_{n_k}), \text{end}\, u) \\ &\quad \leq d((x_{n_k}, \beta), \text{end}\, u) + \varepsilon/2 \\ &\quad \leq H([u_{n_k}]_\beta, [u]_\beta) + \varepsilon/2 \end{aligned} \tag{3}$$

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Note that $H([u_{n_k}]_\beta, [u]_\beta) \to 0$, thus (2) contradicts (3). So the supposition is false.

Suppose $H^*(\operatorname{end} u, \operatorname{end} u_{n_k}) > \varepsilon$. Then similarly we can derive a contradiction.

□

Remark 4.2. Fan (Lemma 1 in [4]) proved a result of Theorem 4.1 type.

Theorem 4.3. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F_{USC}(X)$. Suppose that $H_{\text{end}}(u_n, u) \to 0$. Then $H([u_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in (0, 1) \setminus P_0(u)$*

Proof. Let $\alpha \in (0, 1) \setminus P_0(u)$. Given $\varepsilon > 0$. Then there exists a $\delta(\alpha, \varepsilon) \in (0, \varepsilon/2)$ such that $[\alpha - \delta, \alpha + \delta] \subset [0, 1]$ and

$$H([u]_\beta, [u]_\alpha) < \varepsilon/2 \tag{4}$$

for all $\beta \in [\alpha - \delta, \alpha + \delta]$.

From $H_{\text{end}}(u_n, u) \to 0$, there exists an $N(\delta)$ such that

$$H_{\text{end}}(u_n, u) < \delta \tag{5}$$

for all $n \geq N$. Thus

$$H^*([u_n]_\alpha, [u]_{\alpha-\delta}) < \delta < \varepsilon/2.$$

So, for each $n \geq N$,

$$\begin{aligned} &H^*([u_n]_\alpha, [u]_\alpha) \\ &\leq H^*([u_n]_\alpha, [u]_{\alpha-\delta}) + H([u]_\alpha, [u]_{\alpha-\delta}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \tag{6}$$

Similarly, it follows from (5) that

$$H^*([u]_{\alpha+\delta}, [u_n]_\alpha) < \delta < \varepsilon/2,$$

and then, for each $n \geq N$,

$$\begin{aligned} &H^*([u]_\alpha, [u_n]_\alpha) \\ &\leq H([u]_\alpha, [u]_{\alpha+\delta}) + H^*([u]_{\alpha+\delta}, [u_n]_\alpha) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{7}$$

Combined with (6) and (7),

$$H([u]_\alpha, [u_n]_\alpha) \to 0.$$

□

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The following theorem gives the level decomposition property of H_{end} on $F_{USCG}(X)$.

Theorem 4.4. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F_{USCG}(X)$. Then the following statements are equivalent.*

(i) $H_{\text{end}}(u_n, u) \to 0$

(ii) $H([u_n]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0$

(iii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *for all* $\alpha \in (0,1) \setminus P_0(u)$

(iv) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *when* $\alpha \in P$, *where* P *is a dense subset of* $(0,1)\backslash P_0(u)$

(v) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *when* $\alpha \in P$, *where* P *is a countable dense subset of* $(0,1)\backslash P_0(u)$

Proof. The desired result follows from Lemma 3.7 and Theorems 4.1 and 4.3. □

Remark 4.5. It can be checked that the level decomposition property of H_{end} convergence need not hold on $F_{USC}(X)$.

5. Characterizations of compactness in $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{send}})$

Based on the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{send}})$.

We use $(\widetilde{X}, \widetilde{d})$ to denote the completion of (X, d). We see (X, d) as a subspace of $(\widetilde{X}, \widetilde{d})$. Let $S \subseteq \widetilde{X}$. The symbol $\overline{\widetilde{S}}$ is used to denote the closure of S in $(\widetilde{X}, \widetilde{d})$.

As defined in Section 2, we have $K(\widetilde{X})$, $C(\widetilde{X})$, $F_{USC}(\widetilde{X})$, $F_{USCG}(\widetilde{X})$, etc. according to $(\widetilde{X}, \widetilde{d})$. For example,

$$F_{USC}(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in C(\widetilde{X}) \text{ for all } \alpha \in [0,1]\},$$
$$F_{USCG}(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in K(\widetilde{X}) \text{ for all } \alpha \in (0,1]\}.$$

If there is no confusion, we also use H to denote the Hausdorff metric on $C(\widetilde{X})$ induced by $\widetilde{d}$. We also use H to denote the Hausdorff metric on

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$C(\widetilde{X}\times[0,1])$ induced by $\overline{\widetilde{d}}$. We also use H_{end} to denote the endograph metric on $F_{USC}(\widetilde{X})$ given by using H on $C(\widetilde{X}\times[0,1])$.

Clearly, the induced metric on $F_{USCG}(X)$ by the H_{end} on $F_{USC}(X)$ is the same as the induced metric on $F_{USCG}(X)$ by the H_{end} on $F_{USC}(\widetilde{X})$.

We see $(F_{USCG}(X), H_{\text{end}})$ as a subspace of $(F_{USCG}(\widetilde{X}), H_{\text{end}})$.

5.1. Characterizations of compactness in $(K(X), H)$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(K(X), H)$. The results in this subsection are basis for contents in the sequel.

Theorem 5.1. *Suppose that (X,d) is complete and that $\{C_n\}$ is a Cauchy sequence in $(K(X),H)$. Let $D_n=\bigcup_{l=1}^{n}C_l$ and $D=\overline{\bigcup_{l=1}^{+\infty}C_l}$. Then $D\in K(X)$ and $H(D_n,D)\to 0$.*

Proof. Note that for $k>j$,

$$H(D_k,D_j)\le \max\{H(C_i,C_j): i=j+1,\dots,k\}.$$

So $\{D_n\}$ is a Cauchy sequence in $(K(X),H)$. From Theorem 2.3, $(K(X),H)$ is complete, and thus $\{D_n\}$ converges to $D\in K(X)$. □

Theorem 5.2. *Suppose that (X,d) is a metric space and that $\mathcal{D}\subseteq K(X)$. Then $\mathcal{D}$ is totally bounded in $(K(X),H)$ is equivalent to $\mathbf{D}=\bigcup\{C: C\in\mathcal{D}\}$ is totally bounded in (X,d).*

Proof. If $\mathcal{D}=\emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D}\neq\emptyset$.

Necessity. To show that $\mathbf{D}$ is totally bounded. We only need to show that each sequence in $\mathbf{D}$ has a Cauchy subsequence.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n\in C_n\in\mathcal{D}$. Since $\mathcal{D}$ is totally bounded, then $\{C_n\}$ has a Cauchy subsequence $\{C_{n_k}\}$. Hence, by Theorem 5.1, $\widetilde{\overline{\bigcup_{k=1}^{+\infty}C_{n_k}}}$ is in $K(\widetilde{X})$. Thus $\{x_n\}$ has a Cauchy subsequence.

Sufficiency. If $\mathbf{D}$ is totally bounded in X, then $\widetilde{\overline{\mathbf{D}}}$ is in $K(\widetilde{X})$. So, by Theorem 2.3, $(K(\widetilde{\overline{\mathbf{D}}}),H)$ is compact, and thus $\mathcal{D}$ is totally bounded.

□

Theorem 5.3. *Suppose that (X,d) is a metric space and that $\mathcal{D}\subseteq K(X)$. Then $\mathcal{D}$ is relatively compact in $(K(X),H)$ is equivalent to $\mathbf{D}=\bigcup\{C: C\in\mathcal{D}\}$ is relatively compact in (X,d).*

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Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$.

Necessity. To show that $\mathbf{D}$ is relatively compact. We only need to show that each sequence in $\mathbf{D}$ has a convergent subsequence in X.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n \in C_n \in \mathcal{D}$. Since $\mathcal{D}$ is relatively compact, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ converges to C in $K(X)$. Hence, by Theorem 5.1, $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$ is in $K(\widetilde{X})$ (In fact, $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$ is in $K(X)$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$, and thus $x \in C \subset X$.

Sufficiency. If $\mathbf{D}$ is relatively compact in X, then $\overline{\mathbf{D}}$ is in $K(X)$, and therefore $(K(\overline{\mathbf{D}}), H)$ is compact. Thus $\mathcal{D} \subset K(\overline{\mathbf{D}})$ is relatively compact in $(K(X), H)$.

□

Lemma 5.4. *Suppose that (X, d) is a metric space and that $\mathcal{D} \subseteq K(X)$. If $\mathcal{D}$ is compact in $(K(X), H)$, then $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d)*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$. To show that $\mathbf{D}$ is compact. We only need to show that each sequence in $\mathbf{D}$ has a subsequence converges to a point in $\mathbf{D}$.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n \in C_n \in \mathcal{D}$. Since $\mathcal{D}$ is compact, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ converges to $C \in \mathcal{D}$. Hence, by Theorem 5.1, $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$ is in $K(\widetilde{X})$ (In fact, $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$ is in $K(\mathbf{D})$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\widetilde{\bigcup_{k=1}^{+\infty} C_{n_k}}}$. Thus $x \in C \subset \mathbf{D}$. □

Remark 5.5. The converse of the implication in Lemma 5.4 does not hold. Let $(X, d) = \mathbb{R}$ and let $\mathcal{D} = \{[0, x] : x \in (0.3, 1]\} \subset K(\mathbb{R})$. Then $\mathbf{D} = [0, 1] \in K(\mathbb{R})$. But $\mathcal{D}$ is not compact in $(K(\mathbb{R}), H)$.

Theorem 5.6. *Suppose that (X, d) is a metric space and that $\mathcal{D} \subseteq K(X)$. Then the following statements are equivalent.*
(i) $\mathcal{D}$ is compact in $(K(X), H)$
(ii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$
(iii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$.

Proof. The desired result follows from Theorem 5.3 and Lemma 5.4. □

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Remark 5.7. After we gave conclusions and their proofs in this section, we find Theorem 5.3 is Proposition 5 in [6]. Since we can't find the proof of Proposition 5, we give our proof here.

5.2. Characterizations of compactness in $(F_{USCG}(X), H_{\text{end}})$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}(X), H_{\text{end}})$.

Suppose that U is a subset of $F_{USC}(X)$ and $\alpha \in [0,1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u\in U}[u]_\alpha$, and
- $U_\alpha := \{[u]_\alpha : u \in U\}$.

Theorem 5.8. *Let U be a subset of $F_{USCG}(X)$. Then U is totally bounded in $(F_{USCG}(X), H_{\text{end}})$ if and only if $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0,1]$*

Proof. ***Necessity***. Suppose that U is totally bounded in $(F_{USCG}(X), H_{\text{end}})$. To show that $U(\alpha)$ is totally bounded in X, we only need to show that each sequence in $U(\alpha)$ has a Cauchy subsequence.

Let $\alpha \in (0,1]$. Given $\{x_n\} \subset U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \ldots$. Then $\{u_n\}$ has a Cauchy subsequence $\{u_{n_k}\}$. So given $\varepsilon \in (0, \alpha)$, there is a $K(\varepsilon) \in \mathbb{N}$ such that

$$H_{\text{end}}(u_{n_l}, u_{n_K}) < \varepsilon$$

for all $l \geq K$. Thus

$$H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon \tag{8}$$

for all $l \geq K$. From (8) and the arbitrariness of ε, $\bigcup_{k=1}^{+\infty}[u_{n_k}]_\alpha$ is totally bounded in X. Thus $\{x_{n_k}\}$ has a Cauchy subsequence, and so does $\{x_n\}$.

Sufficiency. Suppose that $U(\alpha)$ is totally bounded in X for each $\alpha \in (0,1]$. By Theorem 5.2, $U(\alpha)$ is totally bounded in X is equivalent to U_α is totally bounded in $(K(X), H)$. Thus, by Theorem 2.3, we have the following affirmation

- Given $\alpha \in (0,1]$. For each sequence $\{[u_n]_\alpha, n = 1, 2, \ldots\}$ in U_α, it has a subsequence $\{[u_{n_k}]_\alpha, k = 1, 2, \ldots\}$ which converges to $u_\alpha \in K(\widetilde{X})$ with respect to the Hausdorff metric H.

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To prove that U is totally bounded, it suffices to show that each sequence in U has a convergent subsequence in $(F_{USCG}(\widetilde{X}), H_{\text{end}})$. Suppose that $\{u_n\}$ is a sequence in U. Based on the above affirmation and Theorem 4.4, and proceeding similarly to the proof of the "Sufficiency part" of Theorem 7.1 in [10], it can be shown that $\{u_n\}$ has a subsequence $\{v_n\}$ which converges to $v \in F_{USCG}(\widetilde{X})$ with respect to H_{end}.

A sketch of the proof of the existence of $\{v_n\}$ and v is given as follows.

First, we construct a subsequence $\{v_n\}$ of $\{u_n\}$ such that $[v_n]_q$ converges to $u_q \in K(\widetilde{X})$ according to the Hausdorff metric H for all $q \in \mathbb{Q}'$, where $\mathbb{Q}' = \mathbb{Q} \cap (0, 1]$. Then we show that $v \in F_{USCG}(\widetilde{X})$ with $[v]_\alpha = \bigcap_{q<\alpha, q\in\mathbb{Q}'} u_q$ for all $\alpha \in (0, 1]$ satisfies that $H_{\text{end}}(v_n, v) \to 0$. $\square$

Remark 5.9. *Some of the implications in the proofs of this paper are actually the equivalent. For example, in the proof of Theorem 5.8, $U(\alpha)$ is totally bounded in X for each $\alpha \in (0, 1]$ is equivalent to the affirmation after the "$\bullet$"*

Theorem 5.10. *Let U be a subset of $F_{USCG}(X)$. Then U is relatively compact in $(F_{USCG}(X), H_{\text{end}})$ if and only if $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.*

Proof. ***Necessity***. Suppose that U is relatively compact. Given $\alpha \in (0, 1]$. To show that $U(\alpha)$ is relatively compact in X, we only need to show that each sequence in $U(\alpha)$ has a convergent subsequence in X.

Let $\{x_n\}$ be a sequence in $U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \ldots$. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in F_{USCG}(X)$ such that $H_{\text{end}}(u_{n_k}, u) \to 0$. So, by Theorem 4.4, $H([u_{n_k}]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0$, and therefore there is a $\beta \in (0, \alpha)$ such that $H([u_{n_k}]_\beta, [u]_\beta) \to 0$. Hence by Theorem 5.3, $\bigcup_{k=1}^{+\infty} [u_{n_k}]_\beta$ is relatively compact in X. Thus $\{x_{n_k}\}$ has a convergent subsequence in X, and so does $\{x_n\}$.

Sufficiency. Suppose that $U(\alpha)$ is relatively compact in X for each $\alpha \in (0, 1]$. To show that U is relatively compact in $(F_{USCG}(X), H_{\text{end}})$, we only need to show that each sequence in U has a convergent subsequence in $(F_{USCG}(X), H_{\text{end}})$.

By Theorem 5.3, $U(\alpha)$ is relatively compact in X is equivalent to U_α is relatively compact in $K(X)$. Thus, we have the following affirmation

- Given $\alpha \in (0, 1]$. For each sequence $\{[u_n]_\alpha, n = 1, 2, \ldots\}$ in U_α, it has a subsequence $\{[u_{n_k}]_\alpha, k = 1, 2, \ldots\}$ which converges to $u_\alpha \in K(X)$ with respect to the Hausdorff metric H.

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The remaining proof is similar to the corresponding part of the "Sufficiency part" of Theorem 5.8.

We can also prove that U is relatively compact in $(F_{USCG}(X), H_{\text{end}})$ as follows. From the "Sufficiency part" of Theorem 5.8, we know that for each sequence $\{u_n\}$ in U, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ which converges to $v \in F_{USCG}(\widetilde{X})$. From Theorem 4.4 and the above statement after the "•", we thus know that $v \in F_{USCG}(X)$.

□

Theorem 5.11. *Let U be a subset of $F_{USCG}(X)$. Then the following statements are equivalent.*

(i) U is compact in $(F_{USCG}(X), H_{\text{end}})$

(ii) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F_{USCG}(X), H_{\text{end}})$

(iii) $U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F_{USCG}(X), H_{\text{end}})$

Proof. The equivalence of statements (i) and (ii) follows immediately from Theorem 5.10. Obviously statement (iii) implies statement (ii).

Now we prove that statement (i) implies statement (iii). Suppose that U is compact. To show that $U(\alpha)$ is compact, we only need to show that $U(\alpha)$ is closed.

Let $\{x_n\}$ be a sequence in $U(\alpha)$ with $x_n \to x$. Suppose that $x_n \in [u_n]_\alpha$ and $u_n \in U$ for $n = 1, 2, \ldots$. Then there exist subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in U$ such that $H_{\text{end}}(u_{n_k}, u) \to 0$. By Theorems 2.1 and 3.4, we have that $\limsup_{n\to\infty}[u_{n_k}]_\alpha \subseteq [u]_\alpha$. Hence $x \in [u]_\alpha$, and thus $x \in U(\alpha)$.

We can also obtain that $x \in [u]_\alpha$ from the fact that $H([u_{n_k}]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0$.

□

5.3. Characterizations of compactness in $(F_{USCB}(X), H_{\text{send}})$

In this subsection, we give the characterizations of totally bounded sets, and compact sets in $(F_{USCB}(X), H_{\text{send}})$. The characterization of relatively compact sets in $(F_{USCB}(X), H_{\text{send}})$ has already been given in [6].

We introduce $P_{USC}(X)$ and $P_{USCB}(X)$ which are subsets of $X \times [0, 1]$.

$$P_{USC}(X) := \{u \subseteq X \times [0,1] : \langle u\rangle_\alpha = \bigcap_{\beta<\alpha} \langle u\rangle_\beta \text{ for all } \alpha \in (0,1];$$
$$\langle u\rangle_\alpha \in C(X) \text{ for all } \alpha \in [0,1]\},$$

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$$P_{USCB}(X) := \{u \in P_{USC}(X) : \langle u \rangle_\alpha \in K(X) \text{ for all } \alpha \in [0,1]\},$$

where $\langle u \rangle_\alpha := \{x : (x, \alpha) \in u\}$ for $u \subseteq X \times [0,1]$ and $\alpha \in [0,1]$.

It can be checked that if $u \in P_{USC}(X)$ then $u \in C(X \times [0,1])$, and that if $u \in P_{USCB}(X)$ then $u \in K(X \times [0,1])$.

It can also be checked that if $u \in C(X \times [0,1])$, then $\langle u \rangle_\alpha \in C(X)$ for all $\alpha \in [0,1]$, and that if $u \in K(X \times [0,1])$, then $\langle u \rangle_\alpha \in K(X)$ for all $\alpha \in [0,1]$.

So we can write

$$P_{USC}(X) = \{u \in C(X \times [0,1]) : \langle u \rangle_\alpha = \bigcap_{\beta < \alpha} \langle u \rangle_\beta \text{ for all } \alpha \in (0,1]\},$$

$$P_{USCB}(X) = \{u \in P_{USC}(X) : u \in K(X \times [0,1])\}.$$

We can formally define H_{send} and H_{end} on $P_{USC}(X)$

$$H_{\text{send}}(u, v) := H(u, v),$$
$$H_{\text{end}}(u, v) := H(\underline{u}, \underline{v}),$$

where $\underline{u} := u \cup (X \times \{0\})$. Clearly, H_{send} is a metric on $P_{USC}(X)$. However, H_{end} need not be a metric on $P_{USC}(X)$.

Consider the function $f : F_{USC}(X) \to P_{USC}(X)$ given by $f(u) = \text{send}\, u$. Then

- f is an isometric embedding of $(F_{USC}(X), H_{\text{send}})$ in $(P_{USC}(X), H_{\text{send}})$.
- $f|_{F_{USCB}(X)}$ is an isometric embedding of $(F_{USCB}(X), H_{\text{send}})$ in $(P_{USCB}(X), H_{\text{send}})$.

Remark 5.12. Clearly, from the above observation, it is natural to discuss the properties of $(F_{USCB}(X), H_{\text{send}})$ by treating $(F_{USCB}(X), H_{\text{send}})$ as a subspace of $(P_{USCB}(X), H_{\text{send}})$, which is a subspace of $(K(X \times [0,1]), H)$.

Obviously, we can think of each $u \in F_{USCG}(X)$ as its endograph, and we can also discuss the properties of $(F_{USCG}(X), H_{\text{end}})$ by treating $(F_{USCG}(X), H_{\text{end}})$ as a subspace of $(C(X \times [0,1]), H)$.

For $u \in F_{USC}(X)$, we use $\overrightarrow{u}$ to denote $f(u)$.

For $v \in P_{USC}(X)$, we use $\overleftarrow{v}$ to denote $f^{-1}(v')$, where $v' \in f(F_{USC}(X))$ is given by

$$\langle v' \rangle_\alpha = \begin{cases} \langle v \rangle_\alpha, & \alpha \in (0,1], \\ \overline{\cup_{\alpha > 0} \langle v \rangle_\alpha}, & \alpha = 0. \end{cases}$$

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Theorem 5.13. *Let u be an element of $P_{USC}(X)$ and $\{u_n\}$ a sequence in $P_{USC}(X)$. Then $H_{\text{send}}(u_n, u) \to 0$ iff $H_{\text{end}}(u_n, u) \to 0$ and $H([u_n]_0, [u]_0) \to 0$*

Proof. Let u, v in $P_{USC}(X)$. Then

$$H_{\text{end}}(u, v) \leq H_{\text{send}}(u, v), \tag{9}$$

$$H([u]_0, [v]_0) \leq H_{\text{send}}(u, v). \tag{10}$$

Thus $H_{\text{send}}(u_n, u) \to 0$ implies that $H_{\text{end}}(u_n, u) \to 0$ and $H([u_n]_0, [u]_0) \to 0$.

Let $(x, \alpha) \in \text{send}\, u$. Clearly, $d((x, \alpha), \text{send}\, v) \leq \alpha + d(x, [v]_0)$. If $\alpha > d((x, \alpha), \text{end}\, v)$, then $d((x, \alpha), \text{send}\, v) = d((x, \alpha), \text{end}\, v)$. Hence

$$d((x, \alpha), \text{send}\, v) \leq \begin{cases} H_{\text{end}}(u, v), & \alpha > H_{\text{end}}(u, v), \\ \alpha + H([u]_0, [v]_0), & \alpha \in [0, 1]. \end{cases}$$

Thus for $u, v \in P_{USC}(X)$ with $H_{\text{end}}(u, v) < 1$

$$H_{\text{send}}(u, v) \leq H_{\text{end}}(u, v) + H([u]_0, [v]_0). \tag{11}$$

(The "=" can be obtained in (11)). So $H_{\text{end}}(u_n, u) \to 0$ and $H([u_n]_0, [u]_0) \to 0$ imply that $H_{\text{send}}(u_n, u) \to 0$.

□

Theorem 5.14. *Suppose that U is a subset of $F_{USCB}(X)$. Then U is totally bounded in $(F_{USCB}(X), H_{\text{send}})$ if and only if $U(0)$ is totally bounded in (X, d).*

Proof. ***Necessity***. Suppose that U is totally bounded. By (10), U_0 is totally bounded in $(K(X), H)$. From Theorem 5.2, this is equivalent to $U(0)$ is totally bounded in (X, d).

Sufficiency. Suppose that $U(0)$ is totally bounded. Then $U(\alpha)$ is totally bounded for each $\alpha \in [0, 1]$. To show that U is totally bounded in $(F_{USCB}(X), H_{\text{send}})$, we only need to prove that each sequence in U has a Cauchy subsequence with respect to H_{send}.

Let $\{u_n\}$ be a sequence in U. Then by Theorem 5.8, $\{u_n\}$ has a Cauchy subsequence $\{v_n\}$ in $(F_{USCB}(X), H_{\text{end}})$. From Theorem 5.2, $\{v_n\}$ has a subsequence $\{w_n\}$ such that $\{[w_n]_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus by (11), $\{w_n\}$ is a Cauchy sequence in $(F_{USCB}(X), H_{\text{send}})$.

□

- $u \in F_{USC}(X)$ is said to be right-continuous at 0 if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $H([u]_\delta, [u]_0) < \varepsilon$.

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- $U \subset F_{USC}(X)$ is said to be equi-right-continuous at 0 if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $H([u]_\delta, [u]_0) < \varepsilon$ for all $u \in U$.

By Lemma 3.6, for each $u \in F_{USCB}(X)$, u is right-continuous at 0.

Theorem 5.15. *[6] Suppose that U is a subset of $F_{USCB}(X)$. Then U is relatively compact in $(F_{USCB}(X), H_{\text{send}})$ if and only if $U(0)$ is relatively compact in X and U is equi-right-continuous at 0.*

Theorem 5.16. *Suppose that U is a subset of $F_{USCB}(X)$. Then the following statements are equivalent.*

(i) U is compact in $(F_{USCB}(X), H_{\text{send}})$

(ii) U is closed in $(F_{USCB}(X), H_{\text{send}})$, $U(0)$ is relatively compact in X and U is equi-right-continuous at 0

(iii) U is closed in $(F_{USCB}(X), H_{\text{send}})$, $U(0)$ is compact in X and U is equi-right-continuous at 0

Proof. The desired result follows immediately from Theorems 5.6 and 5.15. □

6. Completions of $(F_{USCB}(X), H_{\text{send}})$ and $(F_{USCG}(X), H_{\text{end}})$

In this section, we show that $(P_{USCB}(\widetilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}(X), H_{\text{send}})$. We also show that $(F_{USCG}(\widetilde{X}), H_{\text{end}})$ is a completion of $(F_{USCB}(X), H_{\text{end}})$, and thus a completion of $(F_{USCG}(X), H_{\text{end}})$.

Theorem 6.1. *Let (X, d) be a metric space. Then the following statements are equivalent.*

(i) (X, d) is complete.

(ii) $(F_{USCG}(X), H_{\text{end}})$ is complete.

Proof. (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence of $(F_{USCG}(X), H_{\text{end}})$. Then $U = \{u_n, n = 1, 2, \ldots\}$ is total bounded in $(F_{USCG}(X), H_{\text{end}})$. So from the proof the sufficiency part of Theorem 5.8, we know that $\{u_n\}$ has a convergent subsequence in $(F_{USCG}(X), H_{\text{end}})$, and thus $\{u_n\}$ is convergent in $(F_{USCG}(X), H_{\text{end}})$.

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(ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a Cauchy sequence in X. Note that $H_{\text{end}}(\widehat{x}, \widehat{y}) = \min\{d(x,y), 1\}$ for $x, y \in X$. Then $\{\widehat{x_n}\}$ is a Cauchy sequence in $(F_{USCG}(X), H_{\text{end}})$, and therefore $\{\widehat{x_n}\}$ converges to $u \in F_{USCG}(X)$. Thus there exists an $x \in X$ such that $[u]_\alpha = \{x\}$ for all $\alpha \in [0,1]$ (i.e. $u = \widehat{x}$) and $d(x_n, x) \to 0$.

It is easy to see that "(ii) $\Rightarrow$ (i)" can also be proved as follows

(X, d) is complete if and only if (X, d^*) is complete, where $d^*(x,y) = \min\{d(x,y), 1\}$ for $x, y \in X$. Note that $H_{\text{end}}(\widehat{x}, \widehat{y}) = d^*(x,y)$. So the desired result follows from the fact that (X, d^*) is isometric to the closed subspace $(\widehat{X}, H_{\text{end}})$ of $(F_{USCG}(X), H_{\text{end}})$, where $\widehat{X} := \{\widehat{x} : x \in X\}$.

□

Even if (X, d) is complete, $(F_{USCB}(X), H_{\text{send}})$ need not be complete. In fact, $(F_{USCB}(X), H_{\text{send}})$ is complete if and only if X has only one element. We have the following conclusions

Theorem 6.2. *Let (X, d) be a metric space. Then the following statements are equivalent.*

(i) X is complete.

(ii) $(P_{USCB}(X), H_{\text{send}})$ is complete.

Proof. (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence in $(P_{USCB}(X), H_{\text{send}})$. Note that for each $u, v \in P_{USCB}(X)$, $H_{\text{end}}(u, v) = H_{\text{end}}(\overleftarrow{u}, \overleftarrow{v})$. So by (9), $\{\overleftarrow{u_n}\}$ is a Cauchy sequence in $(F_{USCG}(X), H_{\text{end}})$. From Theorem 6.1, $\{\overleftarrow{u_n}\}$ converges to $u \in F_{USCG}(X)$.

By (10), $\{\langle u_n \rangle_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus $\{\langle u_n \rangle_0\}$ converges to $u_0 \in K(X)$.

Set $w \in P_{USCB}(X)$ given by

$$\langle w \rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha > 0, \\ u_0, & \alpha = 0. \end{cases}$$

Thus from Theorem 5.13, u_n converges to w in $(P_{USCB}(X), H_{\text{send}})$.

(ii) $\Rightarrow$ (i). Note that $d(x,y) = H_{\text{send}}(\widehat{x}, \widehat{y})$. So the desired result follows from the fact that (X, d) is isometric to a closed subspace of $(P_{USCB}(X), H_{\text{send}})$.

□

Theorem 6.3. *$(P_{USCB}(\widetilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}(X), H_{\text{send}})$.*

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Proof. From Theorem 6.2, $(P_{USCB}(\widetilde{X}), H_{\text{send}})$ is complete. To show that $(P_{USCB}(\widetilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}(X), H_{\text{send}})$, we only need to show that for each $u \in P_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there is a $w \in F_{USCB}(X)$ such that $H_{\text{send}}(u, \overrightarrow{w}) \leq \varepsilon$. To show this is equivalent to show the following affirmations (a) and (b)

(a) For each $u \in P_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there exists a $v \in F_{USCB}(\widetilde{X})$ such that $H_{\text{send}}(u, \overrightarrow{v}) \leq \varepsilon$

(b) For each $v \in F_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there exists a $w \in F_{USCB}(X)$ such that $H_{\text{send}}(v, w) \leq \varepsilon$

Let $u \in P_{USCB}(\widetilde{X})$. Define $u_\varepsilon \in F_{USCB}(\widetilde{X})$, $\varepsilon > 0$, given by

$$[u_\varepsilon]_\alpha = \begin{cases} \langle u \rangle_\alpha, & \alpha \in (\varepsilon, 1], \\ \langle u \rangle_0, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\text{send}}(u, \overrightarrow{u_\varepsilon}) \leq \varepsilon$. So affirmation (a) is proved.

Let $v \in F_{USCB}(\widetilde{X})$. We can choose a finite subset C_0 of X such that $H(C_0, [v]_0) < \varepsilon$. Define

$$C_\alpha := \{x \in C_0 : d(x, [v]_\alpha) \leq \varepsilon\}, \ \alpha \in (0, 1]. \tag{12}$$

We affirm that $\{C_\alpha : \alpha \in [0, 1]\}$ has the following properties
(i) $C_\alpha \neq \emptyset$ for all $\alpha \in [0, 1]$.
(ii) $H(C_\alpha, [v]_\alpha) \leq \varepsilon$ for all $\alpha \in [0, 1]$.
(iii) $C_\alpha = \bigcap_{\beta < \alpha} C_\beta$ for all $\alpha \in (0, 1]$.
(iv) $C_0 = \bigcup_{\alpha > 0} C_\alpha = \overline{\bigcup_{\alpha > 0} C_\alpha}$.

For each $y \in [v]_\alpha$, there exists $z_y \in C_0$ such that $d(y, z_y) = d(y, C_0) < \varepsilon$. Hence $z_y \in C_\alpha$ and thus $C_\alpha \neq \emptyset$. So (i) is true.

To show (ii), we only need to show that $H(C_\alpha, [v]_\alpha) \leq \varepsilon$ for $0 < \alpha \leq 1$. Let $\alpha \in (0, 1]$. From (12), $H^*(C_\alpha, [v]_\alpha) \leq \varepsilon$. In the following, we show that $H^*([v]_\alpha, C_\alpha) < \varepsilon$. In fact, for each $y \in [v]_\alpha$, $d(y, C_\alpha) \leq d(y, z_y) = d(y, C_0)$ (hence $d(y, C_\alpha) = d(y, C_0)$). Thus $H^*([v]_\alpha, C_\alpha) \leq H([v]_0, C_0) < \varepsilon$. So (ii) is proved.

Let $\alpha \in (0, 1]$. Clearly $C_\alpha \subseteq \bigcap_{\beta < \alpha} C_\beta$. By Lemma 3.6, $\lim_{\beta \to \alpha -} H([v]_\alpha, [v]_\beta) = 0$. So for each $x \in X$, $d(x, [v]_\alpha) = \lim_{\beta \to \alpha -} d(x, [v]_\beta)$, and hence $C_\alpha \supseteq \bigcap_{\beta < \alpha} C_\beta$. Thus $C_\alpha = \bigcap_{\beta < \alpha} C_\beta$. So (iii) is true.

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Let $x \in C_0$. Then $d(x, [v]_0) < \varepsilon$. Since $[v]_0 = \overline{\cup_{\alpha>0}[v]_\alpha}$, there exists $\alpha > 0$ such that $d(x, [v]_\alpha) < \varepsilon$ (in fact, for each $x \in X$, $d(x, [v]_0) = \inf_{\alpha>0} d(x, [v]_\alpha)$), and thus $x \in C_\alpha$. So (iv) is proved.

Set $w \in F(X)$ given by $[w]_\alpha = C_\alpha$ for all $\alpha \in [0, 1]$. Then by (i), (iii) and (iv), $w \in F_{USCB}(X)$. From (ii), we have $H_{\text{send}}(v, w) \leq \varepsilon$. So affirmation (b) is proved.

□

Theorem 6.4. *$(F_{USCG}(\widetilde{X}), H_{\text{end}})$ is a completion of $(F_{USCB}(X), H_{\text{end}})$.*

Proof. From Theorem 6.1, affirmation (b) in the proof of Theorem 6.3 and (9), we only need to show that for each $u \in F_{USCG}(\widetilde{X})$ and each $\varepsilon > 0$, there is a $v \in F_{USCB}(\widetilde{X})$ such that $H_{\text{end}}(u, v) \leq \varepsilon$.

Let $u \in F_{USCG}(\widetilde{X})$. Define $u^\varepsilon \in F_{USCB}(\widetilde{X})$, $\varepsilon > 0$, given by

$$[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\text{end}}(u, u^\varepsilon) \leq \varepsilon$.

□

Corollary 6.5. *$(F_{USCG}(\widetilde{X}), H_{\text{end}})$ is a completion of $(F_{USCG}(X), H_{\text{end}})$.*

Proof. Since $F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USCG}(\widetilde{X})$, the desired result follows from Theorem 6.4.

□

7. Conclusions

In this paper, we point out some elementary relationship among Γ-convergence, H_{end} convergence and H_{send} convergence, and give level characterizations of Γ-convergence and H_{end} convergence on $F_{USC}(X)$.

Based on above results, we discuss characterizations of compactness and completions of two kinds of fuzzy set spaces $(F_{USCG}(X), H_{\text{end}})$ and $(F_{USCB}(X), H_{\text{send}})$, respectively.

In [10], we consider the properties and relationships of Γ-convergence, H_{end} convergence and H_{send} convergence when $X = \mathbb{R}^m$. Some results in this paper improve the corresponding results in [10].

The results in this paper have potential applications in fuzzy set research involving these three convergence structures.

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